Electrostatic potential above a unifomly charged conducting plane deformed to include a hemispherical cup

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The electrostatic potential is calculated above a uniformly charged conducting plane that has been deformed to include a hemispherical cup centered at the origin. The charge density on the surface is obtained.

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A standard problem in electrostatics is as follows: "An infinite conducting sheet in the *xy* plane carries a uniform charge density σ_0. The sheet is isolated and a hemispherical conducting dome having radius a is placed on the sheet, centered at the origin. What is the potential in the space $r > a$, $\theta < \pi/2$ and what is the charge density on the sheet?" Using the general solution of Laplace's equation with boundary conditions

$$\mathbf{E} \sim \sigma_0/\epsilon_0 \hat{\mathbf{z}} \quad z \sim \infty;$$

$$V = 0, \quad r = a, \ \theta \leq \pi/2, \quad \text{and } \theta = \pi/2, \ r > a,$$

one shows easily that the potential is given by

$$V = (\sigma_0/\epsilon_0)\left[-r + \frac{a^3}{r^2}\right]\cos\theta$$

and the charge density by

$$\sigma = \begin{cases} 3\sigma_0 \cos\theta, & r = a, \ \theta \leq \pi/2 \\ \sigma_0\left(1 - \dfrac{a^3}{r^3}\right), & \theta = \pi/2, \ r > a. \end{cases}$$

Note that the charge density vanishes where the dome meets the plane at $r = a$, $\theta = \pi/2$. It is easy to verify that the change in charge per unit azimuthal angle vanishes, that is,

$$d(\delta q)/d\phi = 3\sigma_0 a^2 \int_0^{\pi/2} \cos\theta \sin\theta d\theta + \sigma_0 \int_a^\infty \left(1 - \frac{a^3}{r^3}\right) r dr - \sigma_0 \int_0^\infty r dr = 0.$$

The potential is that of a uniformly charged sheet plus a dipole having dipole moment $\mathbf{p} = 4\pi\sigma_0 a^3 \hat{\mathbf{z}}$ located at the origin.

Suppose, however, that instead of adding a dome above the sheet, one adds a hemispherical conducting shell *below* the sheet and then cuts out a circle having radius a from the sheet to arrive at the surface indicated schematically in Fig. 1. Now the problem is one involving a spherical "cup" rather than a bump. As is seen below, the solution of the potential problem for the cup is no longer trivial and, as might be expected, the results differ qualitatively from the dome problem. The charge density at $r = a$, $\theta = \pi/2$ becomes infinite, but the change in the integrated charge density remains equal to zero. The purpose of this Brief Report is to present a formal solution to this problem. This problem is not only of academic interest. Imperfections in electrodes can be modeled as cups in certain cases. In such situations, the solution presented below can be used to find the modifications in the potential produced by these imperfections. This may be especially important in problems involving ion traps used in laser cooling.

Dimensionless units are introduced where r is measured in units of a, V in units of $(\sigma_0 a/\epsilon_0)$, σ in units of σ_0, charge in units of $\sigma_0 a^2$, and the dipole moment in units of $4\pi\sigma_0 a^3$. Incorporating the boundary conditions $[V(r, \pi/2) = 0, \ r > 1$; $V(1, \theta) = 0, \ \pi/2 \leq \theta \leq \pi$; $V(r = 0)$ is finite$]$, one can write the general solution as

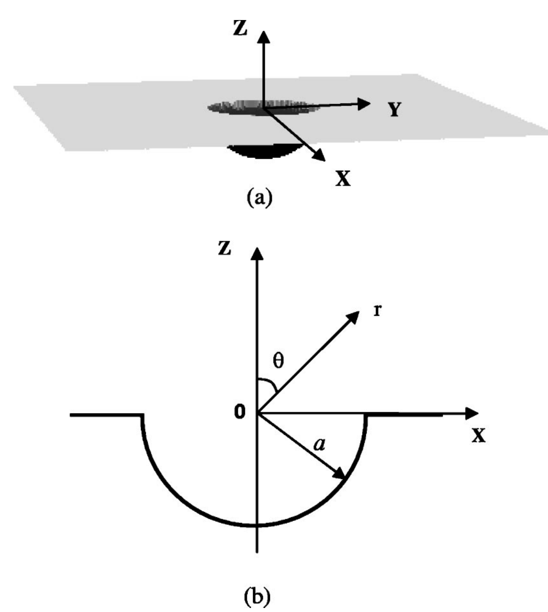

FIG. 1. Bounding surface for the spherical cup problem: (a) three dimensional view, (b) a cut in the *x-z* plane, showing the relevant coordinates.

$$V_{in} = \sum_{n=0}^{\infty} A_n r^n P_n(x), \quad r < 1, \tag{1a}$$

$$V_{out} = -rP_1(x) + \sum_{n=0}^{\infty} \frac{B_{2n+1}}{r^{2n+2}} P_{2n+1}(x), \quad r > 1,\ x > 0, \tag{1b}$$

where $P_n(x)$ is a Legendre polynomial and $x = \cos\theta$. The boundary conditions at $r=1$ require that

$$\sum_{n=0}^{\infty} A_n P_n(x) = \begin{cases} -P_1(x) + \sum_{n=0}^{\infty} B_{2n+1} P_{2n+1}(x), & 0 \leqslant x \leqslant 1 \\ 0, & -1 \leqslant x \leqslant 0. \end{cases} \tag{2}$$

$$\sum_{n=0}^{\infty} nA_n P_n(x) = -P_1(x) - \sum_{n=0}^{\infty} (2n+2) B_{2n+1} P_{2n+1}(x), \quad 0 \leqslant x \leqslant 1. \tag{3}$$

Owing to Dirichlet boundary conditions, one need not specify a boundary condition on the derivatives for $r=1$, $-1 \leqslant x \leqslant 0$.

Using the orthogonality of the Legendre polynomials to solve Eq. (2) for A_n, one finds

$$A_n = \frac{2n+1}{2} \int_0^1 \left[-P_1(x) + \sum_{m=0}^{\infty} B_{2m+1} P_{2m+1}(x) \right] P_n(x) dx. \tag{4}$$

For n odd, this equation yields

$$B_1 = 2A_1 + 1,$$

$$B_{2n+1} = 2A_{2n+1}, \quad n \neq 0. \tag{5}$$

When Eq. (5) is substituted into Eq. (4), one finds, for n even,

$$A_{2n} = \sum_{m=0}^{\infty} F_{2n,2m+1} A_{2m+1}, \tag{6}$$

where

$$F_{\ell,\ell'} = (2\ell+1) \int_0^1 P_\ell(x) P_{\ell'}(x) dx = \frac{(-1)^{(\ell+\ell'-1)/2} (2\ell+1)(\ell-1)!!(\ell')!!}{\left(\frac{\ell}{2}\right)! \left(\frac{\ell'-1}{2}\right)! (\ell'-\ell)(\ell+\ell'+1) 2^{(\ell+\ell'+1)/2}} \tag{7}$$

for $\ell \neq \ell'$. It follows from Eqs. (1), (5), and (6) that, once the odd A_ns are determined, the problem is solved.

To find the A_ns for odd n we use the condition on the derivatives. Multiplying Eq. (3) by $P_{2n+1}(x)$, integrating over x from 0 to 1, and using Eqs. (5)–(7), one obtains

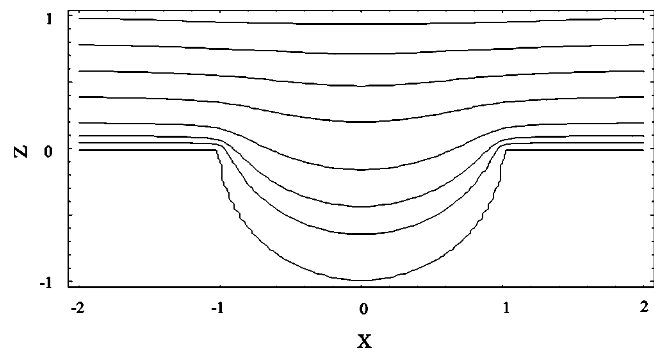

FIG. 2. Equipotential contour plots for the potential above the spherical cup. The equipotentials shown are $V = -0.001$, -0.005, -0.1, -0.2, -0.4, -0.6, -0.8, -1.0.

$$(6n+5) A_{2n+1} = -3\delta_{n,0} - \sum_{m=1}^{\infty} 2m A_{2m} F_{2n+1,2m}.$$

This equation can be rewritten as

$$A_{2n+1} = -\frac{3}{5}\delta_{n,0} - \frac{1}{(6n+5)} \sum_{p=0}^{\infty} G_{2n+1,2p+1} A_{2p+1} \tag{8}$$

with

$$G_{2n+1,2p+1} = \sum_{m=1}^{\infty} 2m F_{2n+1,2m} F_{2m,2p+1}. \tag{9}$$

Formally one can write

$$\mathbf{A}_{odd} = -(3/5)(1 + \tilde{\mathbf{G}})^{-1} \mathbf{S}, \tag{10}$$

where $\mathbf{S}$ is a column matrix having a 1 as its first element and the rest zeroes, while $\tilde{G}_{2n+1,2p+1} = G_{2n+1,2p+1}/(6n+5)$. Equation (10), together with Eqs. (5) and (6), completely specifies the potential.

We evaluated $\mathbf{G}$ as a 151×151 matrix, summing 4000 terms in Eq. (9). The potential was then obtained as a series out to r^{301} for V_{in} and r^{-302} for V_{out}. The explicit expression for $\mathbf{A}_{odd}$ is given in the Appendix. An equipotential plot is shown in Fig. 2. As can be seen, the solution does an excellent job of representing the equipotentials in the region near the "edge." The solutions for V_{in} and V_{out} at $r=1$ agree to within 0.05% for $\theta < \pi/2 - 0.2$ and to within 1.0% for $\theta < \pi/2 - 0.02$, indicating that convergence of the series solution becomes problematic only as one approaches the edge. The charge density on the cup, $\sigma_{cup}(\theta) = \partial V(r,\theta)/\partial r|_{r=1}$ $(\pi/2 \leqslant \theta \leqslant \pi)$, is plotted in Fig. 3(a) and the charge density on the plane $\sigma_{plane}(r) = r^{-1} \partial V(r,\theta)/\partial\theta|_{\theta=\pi/2}$ $(r \geqslant 1)$ in Fig. 3(b). The integrated charge density per unit azimuthal angle on the cup is equal to 0.31195, while the change in the integrated charge density per unit azimuthal angle on the plane, $\int_1^\infty [\sigma_{plane}(r) - 1] r dr$, is equal to 0.17942. As a check of the numerical accuracy one finds that the *change* in the integrated charge per unit azimuthal angle, which should vanish identically, is equal to

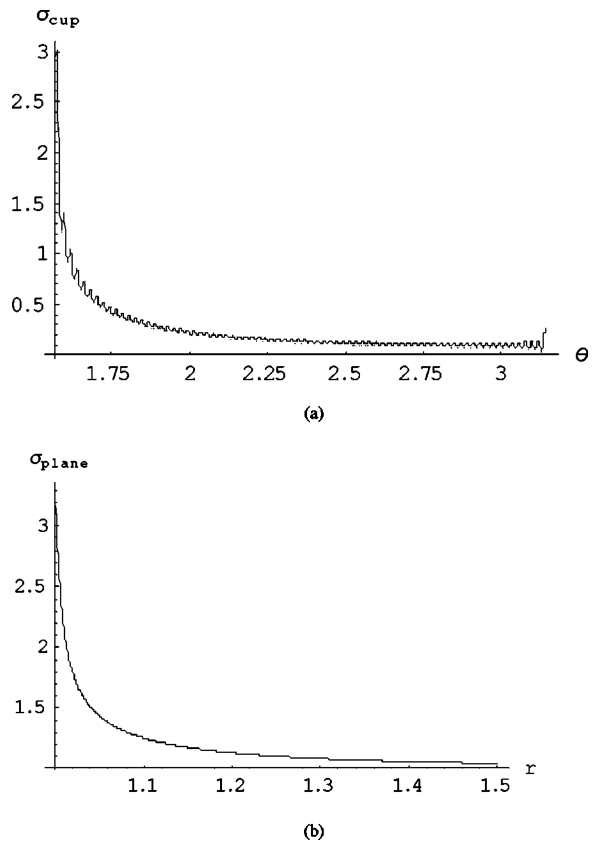

FIG. 3. Charge density (in dimensionless units) for the spherical cup. (a) $\sigma_{cup}(\theta)$ on the spherical cup $(r=1,\pi/2\leqslant\theta\leqslant\pi)$, (b) $\sigma_{plane}(r)$ along the plane $(\theta=\pi/2, r\geqslant 1)$.

$$d(\delta q)/d\phi = \int_{\pi/2}^{\pi} \sin\theta\sigma_{cup}(\theta)d\theta + \int_1^{\infty} r\sigma_{plane}(r)dr - \int_0^{\infty} rdr = -0.00863, \tag{11}$$

for our numerical series solution. At large r the potential is that of a uniformly charged sheet and a dipole having dipole moment $\mathbf{p}=-0.0876\hat{\mathbf{z}}$ centered at the origin.

By comparing our solutions taking different numbers of terms for $\mathbf{A}_{odd}$, we can estimate that the solution for $\sigma_{plane}(r)$ represents an accurate solution for $r\gtrsim 1.012$. In the range $1.012\leqslant r\leqslant 1.015$, one finds that $\sigma_{plane}(r)\sim(r-1)^{-0.264}$. The convergence of the series solution is slow near the edge and it would require thousands of terms to get the correct asymptotic behavior of the charge density as $r\sim 1$. Since the angle between the cup and plane is $\pi/2$, one might expect the charge density to vary as $(r-1)^{-1/3}$ as $r\sim 1$ [1]. In fact, if one fits the charge density as $\sigma_{plane}(r)=a(r-1)^{-1/3}+b(r-1)^{1/3}$ using values of $\sigma_{plane}(r)$ in the range $1.012\leqslant r\leqslant 1.03$, one obtains a best fit for $a=0.753$, $b=0.430$, and the resulting expression, $\sigma_{plane}(r)=0.753(r-1)^{-1/3}+0.430(r-1)^{1/3}$, differs from $\sigma_{plane}(r)$ by no more than 4% in the extended range $1.006\leqslant r\leqslant 1.2$, lending some support to this asymptotic form for the charge density. This form of the charge density also gives some indication of the rate of convergence of the series solution for the potential in the vicinity of the edge. One would expect that, near the edge, the series would converge in a manner similar to the series expansion of $(1-r)^{2/3}$ as $r\sim 1$ [1]. For $r=1-\epsilon$ with $\epsilon\ll 1$, this series converges very slowly and requires approximately $1.74/\epsilon$ terms to achieve an accuracy of 1% [2].

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APPENDIX

The numerically obtained value for $\mathbf{A}_{odd}$ is

$$\mathbf{A}_{odd} = \{-0.5438, 0.02532, -0.01758, 0.01336, -0.01071, 0.008911, -0.007604, 0.006617, -0.005845, 0.005227,$$
$$-0.004721, 0.004299, -0.003943, 0.003638, -0.003374, 0.003144, -0.002942, 0.002762, -0.002602, 0.002458,$$
$$-0.002329, 0.002211, -0.002104, 0.002007, -0.001917, 0.001835, -0.001758, 0.001688, -0.001623, 0.001562,$$
$$-0.001505, 0.001452, -0.001402, 0.001355, -0.001311, 0.001270, -0.001231, 0.001194, -0.001159, 0.001126,$$
$$-0.001095, 0.001065, -0.001037, 0.001010, -0.0009843, 0.0009598, -0.0009363, 0.0009139, -0.0008925, 0.0008720,$$
$$-0.0008523, 0.0008334, -0.0008153, 0.0007979, -0.0007811, 0.0007650, -0.0007495, 0.0007345,$$
$$-0.0007201, 0.0007062, -0.0006928, 0.0006798, -0.0006673, 0.0006552, -0.0006435, 0.0006322,$$
$$-0.0006212, 0.0006105, -0.0006002, 0.0005902, -0.0005805, 0.0005711, -0.0005620, 0.0005531,$$
$$-0.0005445, 0.0005361, -0.0005280, 0.0005201, -0.0005123, 0.0005048, -0.0004975, 0.0004904,$$
$$-0.0004835, 0.0004767, -0.0004701, 0.0004637, -0.0004575, 0.0004513, -0.0004454, 0.0004395,$$
$$-0.0004339, 0.0004283, -0.0004229, 0.0004176, -0.0004124, 0.0004073, -0.0004024, 0.0003975,$$

$-0.0003928, 0.0003881, -0.0003836, 0.0003791, -0.0003748, 0.0003705, -0.0003663, 0.0003622,$
$-0.0003582, 0.0003543, -0.0003504, 0.0003466, -0.0003429, 0.0003393, -0.0003357, 0.0003322,$
$-0.0003288, 0.0003254, -0.0003221, 0.0003189, -0.0003157, 0.0003126, -0.0003095, 0.0003065,$
$-0.0003035, 0.0003006, -0.0002977, 0.0002949, -0.0002921, 0.0002894, -0.0002867, 0.0002841,$
$-0.0002815, 0.0002789, -0.0002764, 0.0002740, -0.0002715, 0.0002691, -0.0002668, 0.0002645,$
$-0.0002622, 0.0002600, -0.0002577, 0.0002556, -0.0002534, 0.0002513, -0.0002492, 0.0002472,$
$-0.0002452, 0.0002432, -0.0002412, 0.0002393, -0.0002374\}.$

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[1] J. D. Jackson, *Classical Electrodynamics*, 3rd Ed. (Wiley, New York, 1999), Sec. 2.11 In the related problem of a cylindrical trough, an analytic solution can be obtained by conformal mapping. In that case the asymptotic form of the charge density on the plane can be shown to vary as $(4/9)\times[2^{1/3}(r-1)^{-1/3}+2^{2/3}(r-1)^{1/3}+O((r-1)^{2/3})]$ as $r\sim 1$ (where r is now the radial distance in dimensionless units of the cylindrical shell radius); however, one must get very close to $r=1$ for the $(r-1)^{-1/3}$ dependence to be seen [at $r=1.01$, $\sigma_{plane}(\text{trough})\sim(r-1)^{-0.3009}$, at $r=1.001$, $\sigma_{plane}(\text{trough})\sim(r-1)^{-0.3316}$ while at $r=1.0001$, $\sigma_{plane}(\text{trough})\sim(r-1)^{-0.3330}$]. The potential along the line from the origin to the edge $(r,\phi=\pi/2)$ varies as $V_{in}(r=1-\epsilon,\phi=\pi/2)\sim -(4/27)^{(1/6)}[\epsilon^{2/3}-\epsilon^{4/3}/2^{2/3}]$.

[2] If $s=(1-r)^{2/3}$ and $s_p=\sum_{n=0}^{p}\binom{2/3}{n}r^n$ is the series approximation to s, one can show that the relative error $y=(s-s_p)/s$ is a function of $x=(1-r)p$ only, and is given by $y\approx b(x)/(x^3+x)$, where $b(x)$ is a slowly varying function of x of order 0.1 in the range $0.01<x<2.5$.